Anand, Bijo S.; **Anil, Arun**; **Changat, Manoj**; **Narasimha-Shenoi, Prasanth G.**; **Ramla, Sabeer S.**

Carathéodory number and exchange number in Δ-convexity. (English) Zbl 08101858

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A (finite) convexity space is a finite set X equipped with a collection $\mathcal{C}$ of subsets of X such that $\emptyset \in \mathcal{C}$ and $\mathcal{C}$ is closed under intersections. The members of $\mathcal{C}$ are said to be convex sets. Given a subset S of X its convex hull, $\langle S\rangle$, is defined to be the intersection of all convex sets containing S. This article considers the notion of Δ-convexity in graphs: given a (finite, simple, undirected) graph G, a subset S of $V(G)$ is Δ-convex if, whenever $v \in V(G)$ forms a triangle with two vertices in S, we have $v \in S$. The collection of Δ-convex sets in G forms a convexity space, which we shall call a Δ-convexity space on G.

Given a convexity space $(X, \mathcal{C})$, several parameters measure its "dimension". A subset S of X is said to be Carathéodory independent if $\langle S\rangle \setminus \bigcup_{a\in S}\langle S\setminus\{a\}\rangle \neq \emptyset$. A subset S of X is said to be exchange independent if S is a singleton or if there exists $p \in S$ such that $\langle S\setminus\{p\}\rangle \setminus \bigcup_{a\in S\setminus\{p\}}\langle S\setminus\{a\}\rangle \neq \emptyset$. The Carathéodory number $c_\Delta(G)$ (resp., exchange number $e_\Delta(G)$) of a convexity space is the maximum size of a Carathéodory independent (resp., exchange independent) set in X. In this article, the authors compute the Carathéodory and exchange numbers for Δ-convexity spaces on chordal graphs, block graphs, and graphs obtained as a box product, $G \,\square\, H$, strong product, $G \boxtimes H$, or lexicographic product, $G \circ H$.

The authors prove that if G is a graph with k triangles, then $c_\Delta(G) \leq k+1$. This bound is tight in general, and moreover for every $n \in \mathbb{N}$ there is a graph G with $c_\Delta(G) = n$. If G is a block graph whose blocks all lie on a single chain, and no block is isomorphic to K_2, then $c_\Delta(G) = \ell + 1$, where ℓ is the number of blocks of G. The authors also prove that if G is a 2-connected chordal graph, then $c_\Delta(G) = 2$. Next, they show that if G and H are graphs such that $c_\Delta(G), c_\Delta(H) > 2$, then $c_\Delta(G \,\square\, H) \geq c_\Delta(G)c_\Delta(H)$, and this lower bound is tight in general. If G and H are connected graphs on at least two vertices, then $c_\Delta(G \boxtimes H) = 2 = c_\Delta(G \circ H)$. Analogous results are also proved for $e_\Delta(G)$.

In Theorems 2.9 and 3.10, the computations of the Carathéodory and exchange numbers, respectively, for block graphs whose blocks do not all lie on a single chain are incorrect. Lemma 2.6(iii) is not true for all graphs, unless $\{u, v\} \notin E(G)$. The proof of Theorem 2.3, which states that $c_\Delta(G) \leq k+1$ when G has k triangles, seems to require further elaboration as the claim in the last paragraph that $\langle S'\rangle = \bigcup_{i=1}^{j}(\langle S'\setminus\{u_i\}\rangle \cup \langle S'\setminus\{v_i\}\rangle)$ does not appear to be entirely straightforward.

Other minor typos are as follows. The notation $\langle S\rangle$ introduced before the statement of Theorem 1.1 for the subgraph of G induced by S conflicts with its usage in the rest of the paper to denote the convex hull of S. In the last line of the proof of Theorem 4.2, the union should be taken over all $(a, b) \in S \setminus \{(g, h_i)\}$. In the statement of Theorem 4.3, read "greater than or equal to two" in place of "greater than two". The last word in the proof of Theorem 4.5 should be "E-dependent".

Reviewer: Brahadeesh Sankarnarayanan (Jodhpur)

MSC:

05C69 Vertex subsets with special properties (dominating sets, independent sets, cliques, etc.)
05C76 Graph operations (line graphs, products, etc.)
05C99 Graph theory
52A01 Axiomatic and generalized convexity

Keywords:

convexity space; Carathéodory number; exchange number; graph products; block graphs

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References:

[1] B. S. Anand, A. Anil, M. Changat, M. C. Dourado, and S. S. Ramla. Computing the hull number in Δ-convexity. Theoretical

Computer Science, 844:217–226, 2020. · Zbl 1468.05047

[2] B. S. Anand, M. Changat, and P. G. Narasimha-Shenoi. Helly and exchange numbers of geodesic and steiner convexities in lexicographic product of graphs. Discrete Mathematics, Algorithms and Applications, 7(04):1550049, 2015. · Zbl 1341.52001

[3] B. S. Anand, P. G. Narasimha-Shenoi, and S. S. Ramla. Δ-convexity number and Δ-number of graphs and graph products. In Conference on Algorithms and Discrete Applied Mathematics, pages 209–218. Springer, 2020. · Zbl 1456.05132

[4] B. S. Anand, U. C. SV, M. Changat, M. C. Dourado, F. H. Nezhad, and P. G. Narasimha-Shenoi. On the carathéodory and exchange numbers of geodetic convexity in graphs. Theoretical Computer Science, 804:46–57, 2020. · Zbl 1442.05040

[5] S. P. Boyd and L. Vandenberghe. Convex Optimization. Cambridge University Press, 2004. · Zbl 1058.90049

[6] C. C. Centeno, M. C. Dourado, L. D. Penso, D. Rautenbach, and J. L. Szwarcfiter. Irreversible conversion of graphs. Theoretical Computer Science, 412(29):3693–3700, 2011. · Zbl 1220.05109

[7] C. C. Centeno, L. D. Penso, D. Rautenbach, and V. G. Pereira de Sa. Geodetic number versus hull number in P⊠-convexity. SIAM Journal on Discrete Mathematics, 27(2):717–731, 2013. · Zbl 1272.05148

[8] M. Changat and J. Mathew. On triangle path convexity in graphs. Discrete Mathematics, 206(1–3):91–95, 1999. · Zbl 0929.05046

[9] M. Changat and J. Mathew. Induced path transit function, monotone and peano axioms. Discrete Mathematics, 286(3):185–194, 2004. · Zbl 1056.05044

[10] M. Changat, H. M. Mulder, and G. Sierksma. Convexities related to path properties on graphs. Discrete Mathematics, 290(2–3):117–131, 2005. · Zbl 1058.05043

[11] E. M. Coelho, M. C. Dourado, D. Rautenbach, and J. L. Szwarcfiter. The carathéodory number of the \textit{P⊠} convexity of chordal graphs. Discrete Applied Mathematics, 172:104–108, 2014. · Zbl 1288.05050

[12] E. Degreef. The convex sum space and direct sum space, simon stevin. Quarterly Journal of Pure and Applied Mathematics, 56:109–119, 1982. · Zbl 0489.52002

[13] J.-P. Doignon, J. R. Reay, and G. Sierksma. A tverberg-type generalization of the helly number of a convexity space. Journal of Geometry, 16:117–125, 1981. · Zbl 0468.52002

[14] M. C. Dourado, L. D. Penso, and D. Rautenbach. On the geodetic hull number of \textit{P⊠}-free graphs. Theoretical Computer Science, 640:52–60, 2016. · Zbl 1345.05017

[15] M. C. Dourado, D. Rautenbach, V. F. Dos Santos, P. M. Schäfer, J. L. Szwarcfiter, and A. Toman. Algorithmic and structural aspects of the \textit{P⊠}-radon number. Annals of Operations Research, 206:75–91, 2013. · Zbl 1270.05062

[16] M. C. Dourado and R. M. Sampaio. Complexity aspects of the triangle path convexity. Discrete Applied Mathematics, 206:39–47, 2016. · Zbl 1335.05045

[17] P. Duchet. Convexity in combinatorial structures. In Proceedings of the 14th Winter School on Abstract Analysis, pages 261–293. Circolo Matematico di Palermo, 1987. · Zbl 0644.52001

[18] P. Duchet. Convex sets in graphs, II. minimal path convexity. Journal of Combinatorial Theory, Series B, 44(3):307–316, 1988. · Zbl 0672.52001

[19] M. Farber. Bridged graphs and geodesic convexity. Discrete Mathematics, 66(3):249–257, 1987. · Zbl 0651.05043

[20] R. E. Jamison. A General Theory of Convexity. University of Washington, 1974.

[21] M. A. Morgana and H. M. Mulder. The induced path convexity, betweenness, and svelte graphs. Discrete Mathematics, 254(1–3):349–370, 2002. · Zbl 1003.05090

[22] I. M. Pelayo. Geodesic Convexity in Graphs. Springer, 2013. · Zbl 1285.05001

[23] J. R. Reay. Generalizations of a Theorem of Carathéodory, number 54. American Mathematical Society, 1965. · Zbl 0137.41503

[24] R. Rockefellar. Convex Analysis. Princeton University Press, 1970. · Zbl 0193.18401

[25] G. Sierksma. Carathéodory and helly-numbers of convex-product-structures. Pacific Journal of Mathematics, 61(1):275–282, 1975. · Zbl 0301.52002

[26] G. Sierksma. Relationships between Carathéodory, Helly, Radon and Exchange Numbers of Convexity Spaces. Rijksuniversiteit, 1976. · Zbl 0319.52003

[27] G. Sierksma. Convexity on unions of sets. Compositio Mathematica, 42(3):391–400, 1980. · Zbl 0454.52001

[28] M. L. van De Vel. Theory of Convex Structures, volume 50. Elsevier, 1993. · Zbl 0785.52001

[29] D. B. West. Introduction to Graph Theory, volume 2. Prentice Hall, Upper Saddle River, 2001.